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An Area-Maximum Edge Length Trade-off for VSLI Layout

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We construct an N-node graph G which has (i) a layout with area $O(N)$ and maximum edge length $O(N^{1/2})$, (ii) a layout with area $O(N^{5/4})$ and maximum edge length $O(N^{1/4})$. We prove for $1 \leqslant f(N) \leqslant O(N^{1/8})$ that any layout for G with area $Nf(N)$ has an edge of length $\Omega(N^{1/2}/f(N) \cdot \log N)$. Hence G has no layout which is optimal with respect to both measures.

1. INTRODUCTION

One of the central problems in VLSI is how to embed a graph in the 2-dimensional grid. Given a graph G, especially the following two questions have found a great deal of attention in past research:

(i) How much area is needed to lay out G in the grid?

(ii) What is the length of the longest wire in any layout for G?

Upper and lower bounds for both questions are known [1–8].

We consider the question, whether one can always optimize both measures simultaneously. The known results suggest that this is in fact possible. The known layouts which achieve the best upper bounds for the maximum edge length simultaneously achieve the best known upper bounds for the area [1]. The graphs, for which the greatest lower bounds for the maximum edge length are known are also graphs with greatest lower bounds for the area [2, 3].

We prove, that we cannot always optimize both measures simultaneously. More precisely, we define an N-node graph G which has a layout with area $O(N)$ and maximum edge length $O(N^{1/2})$ and a layout with area $O(N^{5/4})$ and maximum edge length $O(N^{1/4})$. Furthermore, we prove for $1 \leqslant f(N) \leqslant O(N^{1/8})$ that any layout for G with area $Nf(N)$ has maximum edge length at least $\Omega(N^{1/2}/f(N) \cdot \log N)$.

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2. Definitions

A (*rectangular*) *grid* is a collection of horizontal and vertical lines which are spaced apart at unit intervals.

An *embedding* or *layout* of a graph G in a grid is an assignment of

(i) the nodes of G to intersection points of horizontal and vertical lines in the grid and

(ii) the edges of G to paths along grid lines.

The paths are not allowed to overlap for any distance and are not allowed to cross nodes to which they are not incident.

The *layout area* of an embedding is the product of the number of vertical lines and the number of horizontal lines which contain a node or a path segment of the graph. The *maximum edge length* of a layout is the length of the longest edge in the layout. The *wire area* of a layout is the sum of the individual edge lengths in the layout. Note that layout area $\geqslant$ wire area.

For simplicity of the description, let n be a square root. The graph G_n, for which we prove the tradeoff is constructed by

(i) An $n \times n$ mesh. The rows are numbered by $0, 1, \ldots, n-1$.

(ii) Each node on row 0 is connected with the left leaf of an n-node complete binary tree. The trees are pairwise disjoint.

(iii) There is an additional node for every $\log n$ columns on the rows $0, 2\sqrt{n}-1, 4\sqrt{n}-1, \ldots,$ of the $n \times n$ mesh.

The additional nodes in the same column are connected by a complete binary tree. We denote the additional nodes by leaf-nodes of the $n \times n$ mesh and the other nodes of the $n \times n$ mesh by non-leaf-nodes. Figure 1 illustrates the graph G_n. Note, that G_n has $N = 2n^2 + 2n^{1.5}/\log n - n/\log n$ nodes.

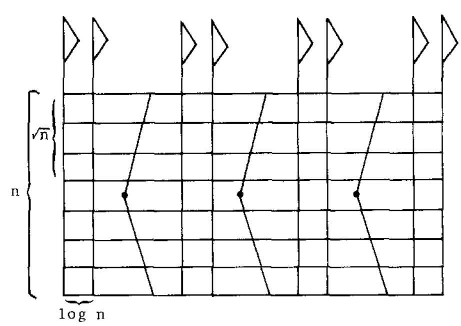

FIG. 1. The graph G_n.

3. The Trade-off

First we prove two upper bounds.

THEOREM 1. (i) *There exists a layout E_1 for G_n with area $O(n^2)$ and maximum edge length $O(n)$.*

(ii) *There exists a layout E_2 for G_n with area $O(n^{2.5})$ and maximum edge length $O(\sqrt{n})$.*

Proof. (i) We embed the $n \times n$ mesh with its $n/\log n$ trees in the obvious way in area $2n^2$. Obviously no edge is longer than $O(n)$. For the layout of an n-node complete binary tree, we use the layout of [6] in a square with side length $c \cdot \sqrt{n}$, for a constant c and with maximum edge length $O(\sqrt{n}/\log n)$. The n squares are grouped in $\sqrt{n}$ blocks of $\sqrt{n}$ squares above row 0. Each tree is connected with its node on row 0, such that no edge is longer than $O(n)$. Totally, we use $O(n^2)$ area and have maximum edge length $O(n)$. Figure 2 illustrates the layout E_1 for G_n.

(ii) In a layout E for G_n denote the length of the embedding of row i (of the modified $n \times n$ mesh) by *length of the row* i (with respect to the layout E).

For embedding G_n with maximum edge length $O(\sqrt{n})$, we have two goals:

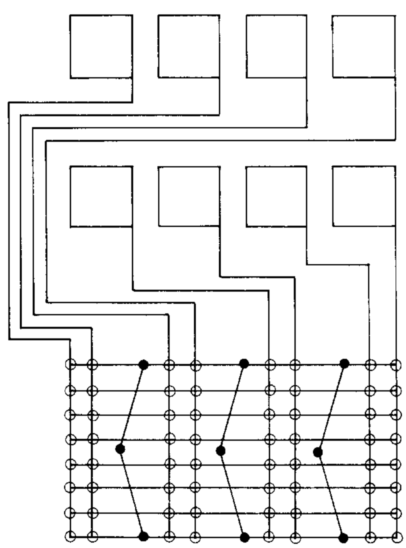

FIG. 2. The layout E_1 for G_n.

(1) Let $c \cdot \sqrt{n}$ be the side length of the layout of the n-node complete binary tree as described in [6]. We will stretch the length of row 0 to $cn^{1.5}$. Then the n complete binary trees can be embedded side by side above row 0 such that for the connection of the trees with row 0 only short edges are necessary.

(2) We embed the $n \times n$ mesh such that the leaf-nodes corresponding to the same tree are not too far apart from each other. Then we can embed the mesh-trees with edges not longer than $O(\sqrt{n})$.

For reaching these goals first we stretch the $n \times n$ mesh in both directions by the factor $\max\{2\sqrt{n}, c\sqrt{n}\}$. Goal (1) is reached.

Then we fold the $n \times n$ mesh $\sqrt{n}$-times, such that goal (2) is reached. The mesh-trees are embedded in the obvious way. For details of the layout, see Fig. 3. Note that each edge has length at most $O(\sqrt{n})$ and the area is $O(n^{2.5})$. ∎

Next we prove the lower bound. For proving the lower bound, the following concept is useful: Consider the area, which is touched by a circle with radius r by moving the center of the circle along a line of length l, beginning at one end point of the line and ending at the other end point.

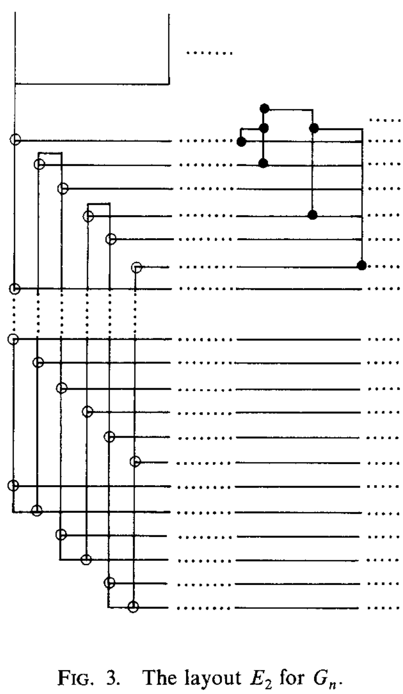

FIG. 3. The layout E_2 for G_n.

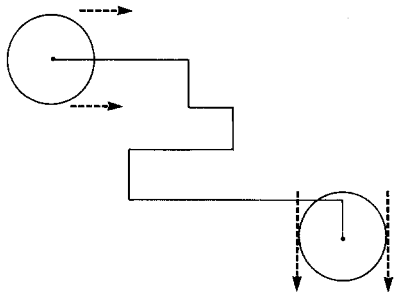

FIG. 4. An (l, r)-area.

We denote such an area by (l, r)-*area*. Figure 4 illustrates an (l, r)-area. It is easy to see, that the area of an (l, r)-area is $\leqslant l \cdot 2r + 2\pi r^2$.

We prove the following theorem:

THEOREM 2. *Let* $1 \leqslant f(n) \leqslant \frac{1}{10} n^{1/4}$. *Then any layout* E *for* G_n *with area* $\leqslant n^2 f(n)$ *has an edge of length at least* $\Omega(n/f(n) \cdot \log n)$.

Proof. The proof divides into two parts. First, we prove that in any layout for G_n with area $\leqslant n^2 f(n)$ and no edge of length $\Omega(n/f(n) \cdot \log n)$ at least $n/2$ of the non-leaf-nodes of row 0 have to lie inside an $(O(nf(n)), O(n/f(n)))$-area. Then we use this fact to prove that an edge of length $\Omega(n/f(n) \cdot \log n)$ must exist.

LEMMA 1. *Let* E *be a layout for* G_n *with*

(i) *area* $\leqslant n^2 f(n)$

(ii) *maximum edge length* $\leqslant n/72 f(n) \cdot \log n$.

Then at least $n/2$ non-leaf-nodes of row 0 have to lie inside an $(l, n/12 f(n))$-area, where $l \leqslant 2nf(n)$.

Proof. We divide the $n \times n$ mesh of G_n in $\sqrt{n}$ blocks as follows:

$$\begin{array}{ll} \text{block } 1\text{:} & \text{row } 0, \ldots, \text{row } \sqrt{n} - 1 \\ \text{block } 2\text{:} & \text{row } \sqrt{n}, \ldots, \text{row } 2\sqrt{n} - 1 \\ \vdots & \\ \text{block } \sqrt{n}\text{:} & \text{row } n - \sqrt{n}, \ldots, \text{row } n - 1. \end{array}$$

Since the layout E has area $\leqslant n^2 f(n)$ there at least $n/2$ rows with length $\leqslant 2nf(n)$. Hence there are at least $\sqrt{n}/2$ blocks with one row of length $\leqslant 2nf(n)$. Each of these $\sqrt{n}/2$ rows of length $\leqslant 2nf(n)$ induce in an obvious way an $(l, n/12f(n))$-area with $l \leqslant 2nf(n)$.

Assume that none of these areas contains $n/2$ of the non-leaf-nodes of row 0. Now we consider one of these $(l, n/12f(n))$-areas fixed. Let row j be the corresponding row and block k be the corresponding block. Let v be an non-leaf-node outside of the $(l, n/12f(n))$-area. Let w be the node on row j, which is in the same column t as the node v. Then we consider the following path P from v to w:

(a) along edges on row 0: from v to the next leaf-node.

(b) along edges in the tree: from this leaf-node to the leaf-node next to block k.

Assume, tnis leaf-node is on row i.

(c) along edges on row i: from this leaf-node to the node u on column t.

(d) along edges on column t: from u to w.

The path P from the node v to the node w is illustrated in Fig. 5.

The length of the path P is at least $n/12f(n)$. At most $3 \log n$ edges of P are on row 0, in the tree or on row i. Since the maximum edge length is $\leqslant n/72f(n) \cdot \log n$ the following holds: The length of the path P in block k is

$$\geqslant \frac{n}{12f(n)} - \frac{(3 \log n)n}{72f(n) \cdot \log n} = \frac{n}{24f(n)}.$$

Hence in block k the path P contributes at least $n/24f(n)$ to the wire area of the layout E.

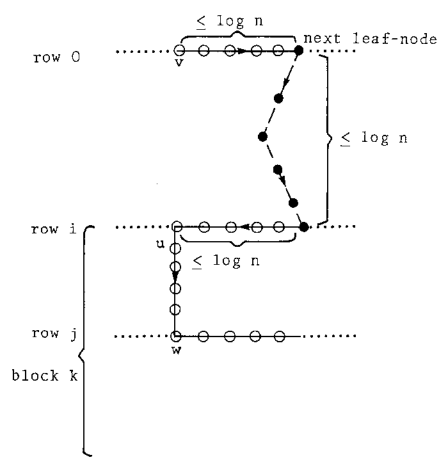

FIG. 5. The path P from the node v to the node w.

For two distinct non-leaf-nodes outside of the $(l, n/12f(n))$-area the corresponding paths are disjoint in block k. Hence we have: Block k contributes at least

$$\frac{n}{2}\cdot\frac{n}{24f(n)}=\frac{n^2}{48f(n)}$$

to the wire area of the layout E.

Since the considered $\sqrt{n}/2$ blocks are pairwise disjoint, we have

$$\begin{aligned}\text{layout area of } E &\geqslant \text{wire area of } E\\ &\geqslant \frac{\sqrt{n}}{2}\cdot\frac{n^2}{48f(n)}=\frac{n^{2.5}}{96f(n)}\\ &> n^2 f(n) \ (\text{since } f(n)\leqslant \tfrac{1}{10}n^{1/4}).\end{aligned}$$

This proves lemma 1.

Now, the theorem follows immediately from the following lemma.

LEMMA 2. *Let E be a layout for G_n with at least $n/2$ non-leaf-nodes of row 0 inside a $(2nf(n), n/12f(n))$-area. Then there exists an edge of length at least $n/24f(n)\cdot\log n$.*

Proof. The non-leaf-nodes of row 0 in the $(2nf(n), n/12f(n))$-area are connected with at least $n/2$ n-node complete binary trees. Consider the smallest $(2nf(n), r)$-area, in which these binary trees can be embedded. It is easy to see that $r > n/6f(n)$. Hence there exists a node v in such a tree, which is at least $n/12f(n)$ units apart from the node w of row 0, which is connected with that tree. On the path from v to w are at most $2\log n$ edges. Hence there exists an edge of length $\geqslant n/24f(n)\cdot\log n$. ▌

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